Vectors and Matrices

Unit 1 · Linear Systems and Matrix Algebra

Concept 1.1
Formulating Systems from Real-World Scenarios

These notes focus on translation only: turning a contextual narrative—such as a pricing model, a resource constraint, a mixture, or an allocation problem—into a system of equations with every variable and constant clearly identified. In this unit, each scenario produces a linear system; solving begins in Concept 1.2.

Learning Objective

Construct a system of linear equations from a contextual narrative, explicitly defining all variables and constant values.

Key Definitions

System of equations. A collection of two or more equations involving the same set of variables that must all hold true at the same time.

Linear equation. An equation in which every variable term consists of a constant (called its coefficient) multiplied by a single variable raised to the first power. Variables are never multiplied together, raised to powers other than 1, or placed in denominators.

Standard form of a linear system. An arrangement in which every equation lists its variable terms on the left-hand side—in the exact same variable order—and its constant term on the right-hand side. Any variable absent from a particular condition has a coefficient of 0.

1.1.1 Example: Pricing Model

> Rowan is repairing their cargo van and purchases 4 spark plugs and 2 quarts of synthetic oil for $34. The next day, Quinn works on a different vehicle and purchases 6 spark plugs and 5 quarts of the same synthetic oil for $73.

Define the unknowns.

The quantities purchased are known constants. The two unit prices are unknown.

- Let p equal the price of one spark plug (in dollars).
- Let q equal the price of one quart of synthetic oil (in dollars).

Both purchases use the same prices, so both equations use the same p and the same q.

Translate the conditions.

The total cost is the sum of quantity times price. The constants 4, 2, 6, and 5 are quantities purchased. The constants 34 and 73 are the totals paid, in dollars.

$$4p + 2q = 34 \quad \leftarrow \textit{Rowan's purchase, in dollars}$$
$$6p + 5q = 73 \quad \leftarrow \textit{Quinn's purchase, in dollars}$$

1.1.2 Example: Resource Constraint

> Each poster uses 3 square feet of vinyl and 2 hours of labor. Each banner uses 6 square feet of vinyl and 3 hours of labor. An order uses exactly 24 square feet of vinyl and 14 hours of labor.

Define the unknowns.

The usage rates and the two totals are known constants. The unknowns are how many of each item are made.

- Let x equal the number of posters made.

- Let y equal the number of banners made.

Translate the conditions.

Write one equation for each resource. In this scenario, each variable's coefficient represents the amount of that resource required per item (3 and 6 square feet per item for vinyl; 2 and 3 hours per item for labor), while the constant on the right-hand side is the total amount of that resource used (24 square feet and 14 hours). The word "exactly" makes each condition an equation; limits stated as "at most" or "at least" produce inequalities, which appear later in the course.

$$3x+6y=24 \quad \leftarrow \textit{square feet of vinyl}$$
$$2x+3y=14 \quad \leftarrow \textit{hours of labor}$$

1.1.3 Example: Mixture

> Colombian coffee costs \$8 per pound, and Sumatran coffee costs \$14 per pound. The café wants 30 pounds of a blend worth \$12 per pound.

Define the unknowns.

The prices per pound and the desired blend are known constants. The unknowns are the amounts of the two coffees.

- Let c equal the number of pounds of Colombian coffee.
- Let s equal the number of pounds of Sumatran coffee.

Translate the conditions.

The known constants are the unit prices (8 and 14 dollars per pound), the desired blend weight (30 pounds), and the target unit price (12 dollars per pound). First, the individual weights add to the 30-pound total weight. Second, the dollar values must balance. Because \$12 per pound is a unit rate rather than a total dollar value, multiply by the weight of the blend first to find the total worth of the mixture: $12 \cdot 30 = 360$ dollars.

$$c+s=30 \quad \leftarrow \textit{total weight of coffee, in pounds}$$
$$8c+14s=360 \quad \leftarrow \textit{total value of the blend, in dollars}$$

1.1.4 Example: Three-Variable Ticket Allocation

> A community theater sells student tickets for \$8, adult tickets for \$15, and senior tickets for \$10. For Friday's performance, the theater sells 100 tickets in total and collects \$1,190 in ticket revenue. Student and senior tickets qualify as discounted admissions, and exactly 50 discounted tickets are sold.

Define the unknowns.

The ticket prices and the three overall totals are known constants. The unknowns are the counts of each ticket type sold.

- Let x equal the number of student tickets sold.
- Let y equal the number of adult tickets sold.
- Let z equal the number of senior tickets sold.

Translate the conditions.

The constants 100, 1,190, and 50 give the total tickets, total dollars collected, and total discounted tickets. The unit prices 8, 15, and 10 (in dollars per ticket) are the coefficients in the revenue equation. Adult tickets are full-price, so y does not participate in the discounted-ticket count; in standard form, $x+z=50$ is aligned using a coefficient of 0 for y.

$$
\begin{aligned}
x + \quad y + \quad z &= 100 && \leftarrow \textit{total tickets sold} \\
8x + 15y + 10z &= 1190 && \leftarrow \textit{total revenue, in dollars} \\
x + \; 0y + \quad z &= 50 && \leftarrow \textit{discounted tickets sold } (x + z = 50)
\end{aligned}
$$

1.1.5 The Formulation Habit

1. Define the unknowns.
   - Assign one variable to each unknown quantity the scenario asks for, and state its unit. A statement such as "Let p equal the price of one spark plug, in dollars" is complete; "Let p equal spark plugs" is not.
2. Separate the conditions.
   - Identify each independent constraint, relationship, or total in the narrative. Each independent condition becomes one equation.
3. Translate and balance units.
   - Express how the known constants and unknown variables interact, ensuring both sides of each equation measure the exact same kind of quantity (dollars equal dollars; hours equal hours). Counts, rates, and totals given in the scenario act as constants in the model, not extra variables.
4. Assemble the system.
   - Group the equations together in a consistent, readable structure so they hold simultaneously. Throughout this unit, our scenarios yield linear systems written in standard form.

1.1.6 Translation Pitfalls

- Multiply unit rates by quantities to express total contributions.
  - When a scenario gives a rate per unit—dollars per spark plug in Example 1.1.1, square feet per poster in Example 1.1.2, or dollars per ticket in Example 1.1.4—multiply that rate by the corresponding quantity so the term represents the total contribution (for example, $4p$ dollars for 4 spark plugs in Example 1.1.1).
- Keep each constant paired with its intended variable.
  - In Rowan's purchase in Example 1.1.1 ($4p + 2q = 34$), 4 counts the spark plugs and 2 counts the quarts of oil; swapping parameters alters the constraint.
- Match units across the equals sign.
  - Both sides of an equation must measure the exact same kind of quantity. In Example 1.1.2, square feet of vinyl equal square feet of vinyl; in Examples 1.1.1, 1.1.3, and 1.1.4, dollars equal dollars.
- Convert target rates into totals before equating values.
  - In the coffee mixture of Example 1.1.3, \$12 is a price per pound, not the total worth of the batch. Multiply by the 30-pound weight first ($12 \cdot 30 = 360$) so both sides of the value equation measure total dollars.
- Preserve variable order and note missing terms.
  - Keeping variables in the same left-to-right order in every equation, and recognizing a missing variable as having coefficient 0, prevents alignment mistakes when translating systems into matrices later in Unit 1. Example 1.1.4 writes the discounted-ticket condition as $x + 0y + z = 50$.