Vectors and Matrices

Unit 1 · Linear Systems and Matrix Algebra

Concept 1.2
Solving via Substitution & Recognizing Algebraic Limitations

In Concept 1.1, we translated real-world narratives into systems of linear equations written in standard form. These notes focus on finding the exact numerical values that satisfy a two-variable system using algebraic substitution and elimination, while examining why unstructured substitution breaks down as systems grow to three or more variables. In Concept 1.3, we will connect these algebraic outcomes to the geometric behavior of lines in the plane.

Learning Objective

Solve two-variable systems algebraically and explain why direct substitution fails to scale efficiently to larger systems (3×3 or higher).

Key Definitions

Solution of a linear system. An ordered list of numbers—such as an ordered pair (x, y) for two variables or an ordered triple (x, y, z) for three variables—that makes every equation in the system a true statement simultaneously when substituted for the variables.

Solution set. The collection of all possible solutions to a given system of equations.

Substitution method. An algebraic procedure in which one equation is solved for one variable in terms of the others, and that expression is substituted into the remaining equations to reduce the number of unknowns by one.

Elimination method. An algebraic procedure in which equations are multiplied by nonzero constants and added together so that a target variable with opposite coefficients cancels out.

1.2.1 Example: Two-Variable Substitution

A campus print shop tracks paper stock across two print runs. The first run uses 1 roll of matte paper and 3 rolls of glossy paper for a total weight of 14 kilograms. The second run uses 2 rolls of matte paper and 5 rolls of glossy paper for a total weight of 25 kilograms. Let x represent the weight of one roll of matte paper (in kilograms) and let y represent the weight of one roll of glossy paper (in kilograms). Find the weight of each roll by solving the system:

$$x + 3y = 14 \quad \leftarrow \textit{first print run weight, in kilograms}$$
$$2x + 5y = 25 \quad \leftarrow \textit{second print run weight, in kilograms}$$

Isolate and substitute to find the first unknown.

Because x has a coefficient of 1 in the first equation, substitution avoids fractions. Subtracting $3y$ from both sides of the first equation isolates x in terms of y, giving $x = 14 - 3y$. Next, replace x in the second equation with the expression $(14 - 3y)$ to produce a single equation involving only y:

$$2(14 - 3y) + 5y = 25 \quad \leftarrow \textit{substitute } 14 - 3y \textit{ for } x \textit{ in equation 2}$$
$$28 - 6y + 5y = 25 \quad \leftarrow \textit{distribute 2 across the parentheses}$$
$$28 - y = 25 \quad \leftarrow \textit{combine like } y \textit{ terms}$$
$$y = 3 \quad \leftarrow \textit{solve for } y$$

Back-substitute and verify the solution.

To find x, substitute $y = 3$ back into the isolated relationship $x = 14 - 3y$, which gives $x = 14 - 3 \cdot 3 = 14 - 9 = 5$. Checking $(x, y) = (5, 3)$ in both original equations confirms $5 + 3 \cdot 3 = 14$ and $2 \cdot 5 + 5 \cdot 3 = 25$.

$$x = 5 \quad \leftarrow \textit{weight of one matte roll, in kilograms}$$
$$y = 3 \quad \leftarrow \textit{weight of one glossy roll, in kilograms}$$

1.2.2 Example: Two-Variable Elimination

> In Concept 1.1, Rowan purchased 4 spark plugs and 2 quarts of synthetic oil for \$34, while Quinn purchased 6 spark plugs and 5 quarts of the same synthetic oil for \$73. Let p equal the price of one spark plug (in dollars) and let q equal the price of one quart of synthetic oil (in dollars). Determine the unit prices by solving the system:

$$4p + 2q = 34 \quad \leftarrow \textit{Rowan's purchase total, in dollars}$$
$$6p + 5q = 73 \quad \leftarrow \textit{Quinn's purchase total, in dollars}$$

Align coefficients and eliminate a variable.

Neither equation has a coefficient of 1, so isolating p directly would introduce fractions such as $\frac{17-q}{2}$ or $\frac{73-5q}{6}$. Instead, we use elimination to cancel p. The least common multiple of the coefficients 4 and 6 is 12. Multiply every term in the first equation by -3 to get $-12p - 6q = -102$, and multiply every term in the second equation by 2 to get $12p + 10q = 146$. Adding these two scaled equations eliminates p:

$$-12p - 6q = -102 \quad \leftarrow \textit{-3 times equation 1}$$
$$12p + 10q = 146 \quad \leftarrow \textit{2 times equation 2}$$
$$4q = 44 \quad \leftarrow \textit{sum of the scaled equations}$$
$$q = 11 \quad \leftarrow \textit{divide by 4 to solve for q}$$

Back-substitute and verify the solution.

Substitute $q = 11$ into the first original equation to obtain $4p + 2 \cdot 11 = 34$, which simplifies to $4p + 22 = 34$, or $4p = 12$, giving $p = 3$. Verifying $(p, q) = (3, 11)$ in the second equation confirms $6 \cdot 3 + 5 \cdot 11 = 18 + 55 = 73$.

$$p = 3 \quad \leftarrow \textit{price of one spark plug, in dollars}$$
$$q = 11 \quad \leftarrow \textit{price of one quart of oil, in dollars}$$

1.2.3 Example: Scaling Breakdown in Three Variables

> Consider a 3×3 linear system in variables x, y, and z where no variable has a coefficient of 1 or -1:

$$2x + 3y - 2z = 14 \quad \leftarrow \textit{first linear constraint}$$
$$3x - 2y + 4z = 9 \quad \leftarrow \textit{second linear constraint}$$
$$4x + 5y + 3z = 38 \quad \leftarrow \textit{third linear constraint}$$

Attempt direct substitution to reduce the system.

To eliminate x via substitution, solve the first equation for x by subtracting $3y - 2z$ and dividing every term by 2, yielding $x = 7 - \frac{3}{2}y + z$. Because the system has three equations, this three-term fractional expression must be substituted into *both* the second and third equations:

$$3\left(7-\frac{3}{2}y+z\right)-2y+4z=9 \quad \leftarrow \textit{substitute for } x \textit{ in equation 2}$$
$$4\left(7-\frac{3}{2}y+z\right)+5y+3z=38 \quad \leftarrow \textit{substitute for } x \textit{ in equation 3}$$

Expanding the first substituted equation gives $21-\frac{9}{2}y+3z-2y+4z=9$, which combines to $-\frac{13}{2}y+7z=-12$, or $13y-14z=24$ after clearing fractions. Expanding the second substituted equation gives $28-6y+4z+5y+3z=38$, which simplifies to $-y+7z=10$. Completing the solution from this reduced 2×2 system yields $y=7z-10$, then $13(7z-10)-14z=24$, or $77z=154$, giving $z=2$, $y=4$, and $x=3$.

$$13y-14z=24 \quad \leftarrow \textit{reduced equation from substituting into equation 2}$$
$$-y+7z=10 \quad \leftarrow \textit{reduced equation from substituting into equation 3}$$
$$(x,y,z)=(3,4,2) \quad \leftarrow \textit{verified ordered triple solution}$$

Analyze why direct substitution fails to scale.

While direct substitution eventually produces $(x,y,z)=(3,4,2)$, tracking why the algebra grew cumbersome reveals three structural limitations for 3×3 and larger $n\times n$ systems:

1. Cascading multi-term substitutions.
   - In an $n\times n$ system, isolating the first variable creates an expression with up to n terms that must be substituted into $n-1$ remaining equations, requiring $(n-1)$ separate distributions just to drop from n variables to $n-1$.
2. Compounding fractional arithmetic.
   - Dividing by leading coefficients introduces fractions at the first stage that must be distributed, given common denominators, and re-divided at every subsequent substitution stage.
3. No standardized bookkeeping structure.
   - Because substituted expressions change shape at every step, there is no fixed grid of coefficients to track—making human arithmetic errors likely and making direct substitution impractical compared to the systematic matrix row-reduction algorithms introduced in Topic 2.

1.2.4 The Algebraic Reduction Habit

1. Isolate or align a single target variable.
   - Inspect the coefficients first: if an equation has a coefficient of 1 or -1, solve for that variable to prepare for substitution; otherwise, multiply one or both equations by constants to create opposite coefficients on a shared variable for elimination.
2. Collapse the system into a smaller problem.
   - Substitute the isolated expression into the remaining equation, or add the aligned equations together, so that one variable vanishes and leaves a single equation in one unknown.
3. Back-substitute and verify the ordered solution.
   - Solve the single-variable equation, substitute that value back into an original equation to find the remaining unknown, and check the pair in every original equation.
4. Audit the algebraic cost as dimension grows.
   - Notice how each substitution step distributes across every remaining equation and variable, causing fractional coefficients and nested algebra to compound rapidly in systems of size 3×3 and larger.

1.2.5 Algebraic Solution Pitfalls

- Substitute an isolated expression into a *different* equation.
  - Plugging an isolated expression back into the very same equation it came from cancels every variable and yields the tautology $0 = 0$ without finding any unknown; in Example 1.2.1, $x = 14 - 3y$ comes from the first equation and must be substituted into the second equation $2x + 5y = 25$.
- Distribute coefficients across every term inside parentheses.
  - When substituting a multi-term expression or scaling an equation for elimination, multiply every term on both sides; in Example 1.2.1, writing $2(14 - 3y)$ as $28 - 3y$ instead of $28 - 6y$ corrupts the coefficient of y.
- Scale both sides of an equation during elimination.
  - A frequent error is multiplying the left-hand side coefficients while leaving the right-hand side constant unchanged; in Example 1.2.2, multiplying $4p + 2q = 34$ by -3 must scale 34 to -102 as well.
- Check the candidate solution in *every* original equation.
  - Verifying a solution pair in only one equation catches back-substitution slips but misses errors made during the reduction step; always test $(p, q) = (3, 11)$ in both $4p + 2q = 34$ and $6p + 5q = 73$ as shown in Example 1.2.2.