Vectors and Matrices

Unit 1 · Linear Systems and Matrix Algebra

Concept 1.2 · Interactive Reading Guide

Solving via Substitution & Recognizing Algebraic Limitations

In Concept 1.1, we translated real-world narratives into systems of linear equations written in standard form. These notes focus on finding the exact numerical values that satisfy a two-variable system using algebraic substitution and elimination, while examining why unstructured substitution breaks down as systems grow to three or more variables. In Concept 1.3, we will connect these algebraic outcomes to the geometric behavior of lines in the plane.

Learning Objective

Solve two-variable systems algebraically and explain why direct substitution fails to scale efficiently to larger systems (3×3 or higher).

Key Definitions

Solution of a linear system.
Write the definition.

Solution set.
Write the definition.

Substitution method.
Write the definition.

Elimination method.
Write the definition.

1.2.1 Example: Two-Variable Substitution

A campus print shop tracks paper stock across two print runs. The first run uses 1 roll of matte paper and 3 rolls of glossy paper for a total weight of 14 kilograms. The second run uses 2 rolls of matte paper and 5 rolls of glossy paper for a total weight of 25 kilograms. Let x represent the weight of one roll of matte paper (in kilograms) and let y represent the weight of one roll of glossy paper (in kilograms). Find the weight of each roll by solving the system:

$$x + 3y = 14 \quad \leftarrow \textit{first print run weight, in kilograms}$$
$$2x + 5y = 25 \quad \leftarrow \textit{second print run weight, in kilograms}$$

Isolate and substitute to find the first unknown.

Workspace: Isolate and substitute to find the first unknown.

Back-substitute and verify the solution.

Workspace: Back-substitute and verify the solution.

1.2.2 Example: Two-Variable Elimination

In Concept 1.1, Rowan purchased 4 spark plugs and 2 quarts of synthetic oil for \$34, while Quinn purchased 6 spark plugs and 5 quarts of the same synthetic oil for \$73. Let p equal the price of one spark plug (in dollars) and let q equal the price of one quart of synthetic oil (in dollars). Determine the unit prices by solving the system:

$$4p + 2q = 34 \quad \leftarrow \textit{Rowan's purchase total, in dollars}$$
$$6p + 5q = 73 \quad \leftarrow \textit{Quinn's purchase total, in dollars}$$

Align coefficients and eliminate a variable.

Workspace: Align coefficients and eliminate a variable.

Back-substitute and verify the solution.

Workspace: Back-substitute and verify the solution.

1.2.3 Example: Scaling Breakdown in Three Variables

Consider a 3×3 linear system in variables x, y, and z where no variable has a coefficient of 1 or -1:

$$
\begin{aligned}
2x + 3y - 2z &= 14 \quad \leftarrow \textit{first linear constraint} \\
3x - 2y + 4z &= \ \ 9 \quad \leftarrow \textit{second linear constraint} \\
4x + 5y + 3z &= 38 \quad \leftarrow \textit{third linear constraint}
\end{aligned}
$$

Attempt direct substitution to reduce the system.

Workspace: Attempt direct substitution to reduce the system.

Analyze why direct substitution fails to scale.

Workspace: Analyze why direct substitution fails to scale.

1.2.4 The Algebraic Reduction Habit

1. Isolate or align a single target variable.
   - Inspect the coefficients first: if an equation has a coefficient of 1 or -1, solve for that variable to prepare for substitution; otherwise, multiply one or both equations by constants to create opposite coefficients on a shared variable for elimination.
2. Collapse the system into a smaller problem.
   - Substitute the isolated expression into the remaining equation, or add the aligned equations together, so that one variable vanishes and leaves a single equation in one unknown.
3. Back-substitute and verify the ordered solution.
   - Solve the single-variable equation, substitute that value back into an original equation to find the remaining unknown, and check the pair in every original equation.
4. Audit the algebraic cost as dimension grows.
   - Notice how each substitution step distributes across every remaining equation and variable, causing fractional coefficients and nested algebra to compound rapidly in systems of size 3×3 and larger.

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1.2.5 Algebraic Solution Pitfalls

- Substitute an isolated expression into a *different* equation.
  - Plugging an isolated expression back into the very same equation it came from cancels every variable and yields the tautology $0 = 0$ without finding any unknown; in Example 1.2.1, $x = 14 - 3y$ comes from the first equation and must be substituted into the second equation $2x + 5y = 25$.
- Distribute coefficients across every term inside parentheses.
  - When substituting a multi-term expression or scaling an equation for elimination, multiply every term on both sides; in Example 1.2.1, writing $2(14 - 3y)$ as $28 - 3y$ instead of $28 - 6y$ corrupts the coefficient of y.
- Scale both sides of an equation during elimination.
  - A frequent error is multiplying the left-hand side coefficients while leaving the right-hand side constant unchanged; in Example 1.2.2, multiplying $4p + 2q = 34$ by -3 must scale 34 to -102 as well.
- Check the candidate solution in *every* original equation.
  - Verifying a solution pair in only one equation catches back-substitution slips but misses errors made during the reduction step; always test $(p, q) = (3, 11)$ in both $4p + 2q = 34$ and $6p + 5q = 73$ as shown in Example 1.2.2.

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Alkimi Sandbox 1: Verify Your Work

A campus snack bar sells muffins and bottled juices. One order of 2 muffins and 3 juices costs \$13. Another order of 4 muffins and 1 juice costs \$11. Let m be the price of one muffin (in dollars) and let j be the price of one juice (in dollars). The scenario gives the system $2m + 3j = 13$ and $4m + j = 11$.

Solve the system by substitution or elimination, state the ordered pair (m, j), and check both original equations. Check your work in Alkimi Sandbox 1 on the Canvas page.

Reduce the system to one unknown.

Workspace: Reduce the system to one unknown.

Back-substitute and verify the solution.

Workspace: Back-substitute and verify the solution.