Vectors and Matrices

Unit 1 · Linear Systems and Matrix Algebra

Concept 1.3
Geometric Interpretation of Solutions
(Intersecting, Parallel, Coincident Lines)

In Concept 1.2, we solved two-variable linear systems using algebraic substitution and elimination. These notes connect those algebraic outcomes—a single coordinate pair, a false statement such as $0 = 5$, or an identity such as $0 = 0$—to the geometric behavior of lines in the Cartesian plane $\mathbb{R}^2$, classifying each system by its consistency and dependency. In Unit 2, we will transition from algebraic equations to matrix representations and systematic row reduction.

Learning Objective

Classify a system as consistent-independent, consistent-dependent, or inconsistent based on both algebraic outcomes (e.g., $0 = 0$ vs. $0 = 5$) and geometric line relationships.

Key Definitions

Consistent-Independent System. A linear system that possesses exactly one unique solution; in $\mathbb{R}^2$, the equations represent intersecting lines that cross at a single point.

Inconsistent System. A linear system that possesses no solutions because its equations impose contradictory conditions; in $\mathbb{R}^2$, the equations represent strictly parallel lines that never intersect.

Consistent-Dependent System. A linear system that possesses infinitely many solutions because one equation is an algebraic redundant multiple of the other; in $\mathbb{R}^2$, the equations represent coincident lines that occupy the exact same set of points.

Key Properties

Consistent-Independent. Solution set: exactly one solution (x, y). Algebraic outcome: unique values $x = x_0$, $y = y_0$. Geometric configuration in $\mathbb{R}^2$: intersecting lines. Slope-intercept condition: $m_1 \neq m_2$.

Inconsistent. Solution set: no solution ($\emptyset$). Algebraic outcome: false statement (e.g., $0 = 5$). Geometric configuration in $\mathbb{R}^2$: strictly parallel lines. Slope-intercept condition: $m_1 = m_2$ and $b_1 \neq b_2$.

Consistent-Dependent. Solution set: infinitely many solutions. Algebraic outcome: true identity (e.g., $0 = 0$). Geometric configuration in $\mathbb{R}^2$: coincident (overlapping) lines. Slope-intercept condition: $m_1 = m_2$ and $b_1 = b_2$.

Figure 1 compares the three line configurations in $\mathbb{R}^2$.

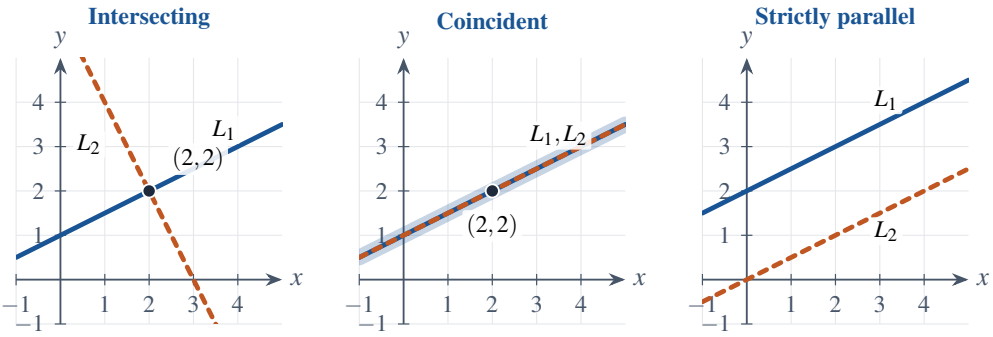


Figure 1: Unequal slopes meet at one point (left): L_1 is solid and L_2 is dashed. Identical slope and intercept describe one line, drawn as a navy stroke under a dashed rust stroke (center). Equal slopes with different intercepts stay strictly parallel (right), again with L_1 solid and L_2 dashed.

1.3.1 Example: Intersecting Lines and Unique Solutions

> Determine the algebraic outcome and geometric line relationship for the following two-variable system in $\mathbb{R}^2$, and classify the system:

$$2x+y=11$$
$$x-y=1$$

Analyze the algebraic outcome and slope-intercept geometry.

Adding the two equations directly eliminates y, yielding $(2x+x)+(y-y)=11+1$, which simplifies to $3x=12$, or $x=4$. Substituting $x=4$ into the second equation gives $4-y=1$, so $y=3$. Algebraically, the system resolves to a single ordered pair $(4,3)$. To verify the geometry, we isolate y in both equations to compare their slopes and y-intercepts. Subtracting $2x$ from both sides of the first equation gives $y=-2x+11$, which has slope $m_1=-2$ and y-intercept $b_1=11$. Subtracting x and multiplying by -1 in the second equation gives $y=x-1$, which has slope $m_2=1$ and y-intercept $b_2=-1$.

Classify the system.

Because the slopes are distinct ($m_1=-2\neq 1=m_2$), the two lines are non-parallel and must cross at a single point in the plane. The unique intersection point $(4,3)$ confirms that the system is consistent and independent:

$$y=-2x+11 \quad \leftarrow \textit{ line 1 with slope } m_1=-2 \textit{ and intercept } b_1=11$$
$$y=x-1 \quad \leftarrow \textit{ line 2 with slope } m_2=1 \textit{ and intercept } b_2=-1$$
$$(x,y)=(4,3) \quad \leftarrow \textit{ intersecting lines: consistent-independent system}$$

Figure 2 plots these lines and their intersection $(4,3)$.

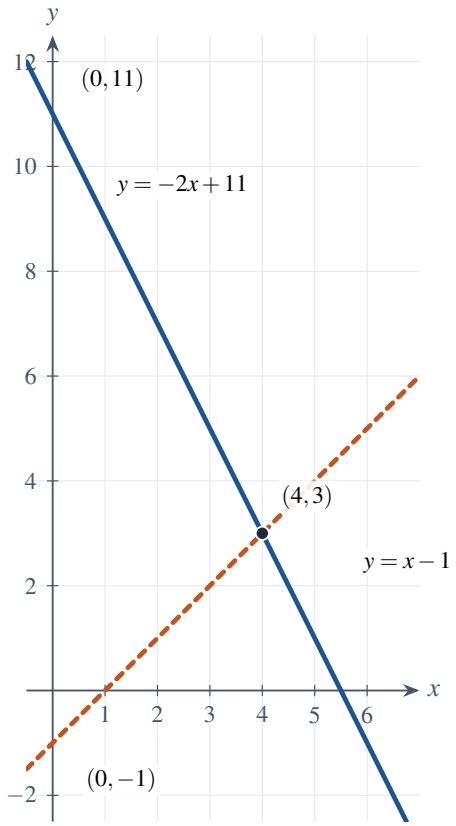


Figure 2: The solid graph of $y = -2x + 11$ and the dashed graph of $y = x - 1$ intersect at the solution $(4,3)$. Both y-intercepts, $(0,11)$ and $(0,-1)$, lie in the window.

1.3.2 Example: Parallel Lines and Algebraic Contradictions

Analyze the following linear system both algebraically and geometrically, and classify the system:

$$\begin{aligned} 3x - y &= 4 \\ 6x - 2y &= 3 \end{aligned}$$

Analyze the algebraic outcome and slope-intercept geometry.

Solving the first equation for y by adding y and subtracting 4 yields $y = 3x - 4$. Substituting $3x - 4$ for y in the second equation produces $6x - 2(3x - 4) = 3$. Distributing -2 across the parentheses gives $6x - 6x + 8 = 3$, where the variable terms cancel completely to leave $8 = 3$. Subtracting 3 from both sides writes this contradiction in standard form as $5 = 0$, or $0 = 5$. To see why the variables vanished into a falsehood, we also write the second equation in slope-intercept form: subtracting $6x$ gives $-2y = -6x + 3$, and dividing by -2 gives $y = 3x - \frac{3}{2}$.

Classify the system.

Both lines share the exact same slope ($m_1 = 3$ and $m_2 = 3$), which means they rise at the same rate across the plane. However, their y-intercepts are different ($b_1 = -4$ versus $b_2 = -\frac{3}{2}$), so the two lines are strictly parallel and separated by a constant vertical distance of $\frac{5}{2}$ units. Because strictly parallel lines have no points in common and the algebraic reduction yields the contradiction $0 = 5$, the system has no solution:

$$\begin{aligned} y &= 3x - 4 && \leftarrow \textit{ line 1 with slope } m_1 = 3 \textit{ and intercept } b_1 = -4 \\ y &= 3x - \frac{3}{2} && \leftarrow \textit{ line 2 with slope } m_2 = 3 \textit{ and intercept } b_2 = -\tfrac{3}{2} \\ 0 &= 5 && \leftarrow \textit{ strictly parallel lines: inconsistent system} \end{aligned}$$

Figure 3 plots these strictly parallel lines.

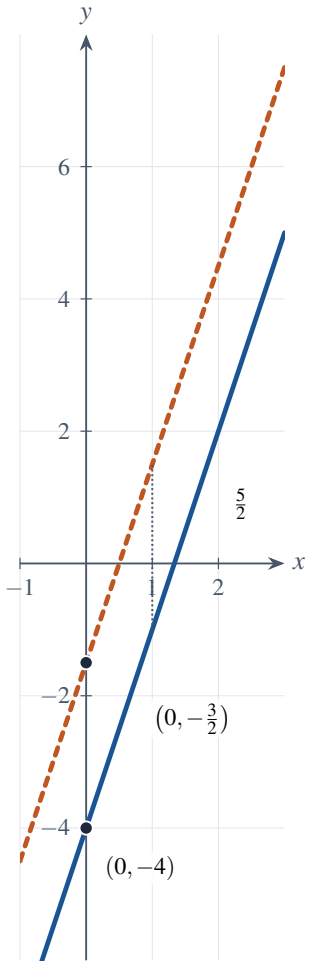


Figure 3: The solid line $y = 3x - 4$ and the dashed line $y = 3x - \frac{3}{2}$ have the same slope, and their y-intercepts are $\frac{5}{2}$ units apart, so the lines never meet.

1.3.3 Example: Coincident Lines and Redundant Constraints

> Elena is calibrating two mixing recipes for a fruit base at a campus café. Let x represent the volume of concentrated juice (in fluid ounces) and let y represent the volume of water (in fluid ounces). The first recipe chart requires $2x + 4y = 12$, while a second production sheet requires $3x + 6y = 18$. Determine the algebraic and geometric relationship between these two constraints and classify the system:

$$2x + 4y = 12$$
$$3x + 6y = 18$$

Analyze the algebraic outcome and slope-intercept geometry.

Dividing the first equation by 2 isolates $x + 2y = 6$, or $x = 6 - 2y$. Substituting $6 - 2y$ for x in the second equation yields $3(6 - 2y) + 6y = 18$. Distributing 3 gives $18 - 6y + 6y = 18$, which simplifies to $18 = 18$, or $0 = 0$. Unlike a contradiction, the identity $0 = 0$ is always true regardless of the value of y. Rewriting both equations in slope-intercept form explains this result: solving $2x + 4y = 12$ for y gives $4y = -2x + 12$, or $y = -\frac{1}{2}x + 3$, and solving $3x + 6y = 18$ for y gives $6y = -3x + 18$, which also simplifies to $y = -\frac{1}{2}x + 3$.

Classify the system.

Both equations have identical slopes ($m_1 = m_2 = -\frac{1}{2}$) and identical y-intercepts ($b_1 = b_2 = 3$). Because the second equation is simply $\frac{3}{2}$ times the first equation, both equations describe the exact same line in $\mathbb{R}^2$. Any point (x, y) that lies on the line $y = -\frac{1}{2}x + 3$—such as $(6, 0)$ or $(2, 2)$—satisfies both equations simultaneously, making the system consistent and dependent:

$$y = -\frac{1}{2}x + 3 \quad \leftarrow \textit{ line 1 with slope } m_1 = -\tfrac{1}{2} \textit{ and intercept } b_1 = 3$$
$$y = -\frac{1}{2}x + 3 \quad \leftarrow \textit{ line 2 with slope } m_2 = -\tfrac{1}{2} \textit{ and intercept } b_2 = 3$$
$$0 = 0 \quad \leftarrow \textit{ coincident lines: consistent-dependent system}$$

Figure 4 plots the shared line together with $(0, 3)$, $(2, 2)$, and $(6, 0)$.

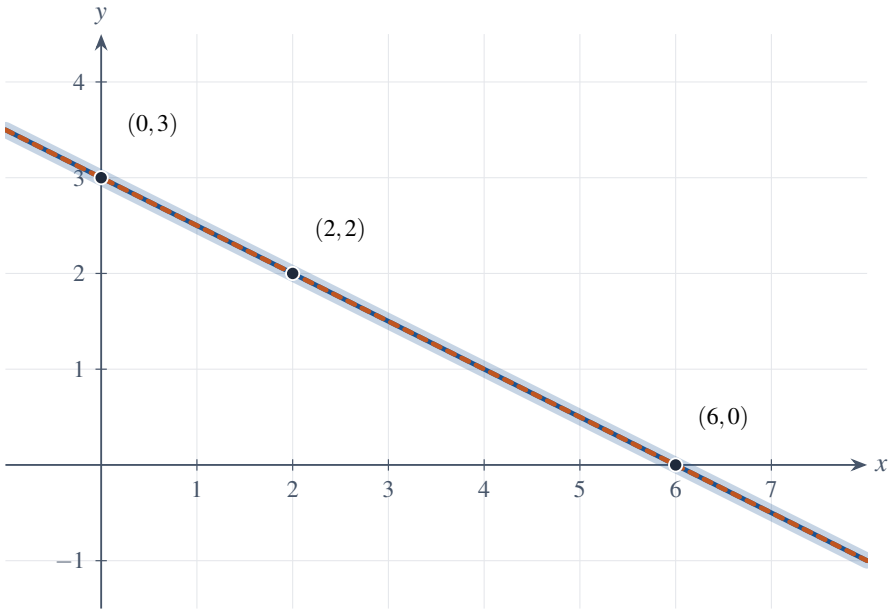


Figure 4: Both equations describe $y = -\frac{1}{2}x + 3$. The intercept $(0, 3)$ and the sample points $(2, 2)$ and $(6, 0)$ lie on that shared line.

1.3.4 The Geometric Classification Habit

When analyzing a mathematical system, translating between algebraic expressions and geometric graphs reveals why a system has one, none, or infinitely many solutions. In $\mathbb{R}^2$, each linear equation defines a straight line whose points (x, y) satisfy that equation, so the solution set of a two-variable system consists of every point that lies on all lines in the system simultaneously.

1. Reduce the system algebraically or rewrite each equation in slope-intercept form.
   - Use substitution or elimination to test whether the variables resolve to a specific pair (x, y) or vanish completely, and rewrite each equation as $y = mx + b$ (or $x = c$ for vertical lines) to identify the slope m and y-intercept b.
2. Interpret the resulting algebraic statement.
   - A specific coordinate pair (x, y) indicates a single shared solution; a false numerical statement such as $0 = 5$ indicates a contradiction with no solutions; a universally true identity such as $0 = 0$ indicates that the equations impose the exact same constraint.
3. Match the slope and intercept relationship to the line configuration.
   - Compare the slopes and intercepts: unequal slopes ($m_1 \neq m_2$) produce intersecting lines, equal slopes with distinct intercepts ($m_1 = m_2$ and $b_1 \neq b_2$) produce strictly parallel lines, and identical slopes with identical intercepts ($m_1 = m_2$ and $b_1 = b_2$) produce coincident lines.

4. State the complete two-part classification.
   – Pair the existence label (consistent or inconsistent) with the uniqueness label (independent or dependent) to classify the system as consistent-independent, inconsistent, or consistent-dependent.

1.3.5 Geometric Classification Pitfalls

- Distinguish between the algebraic identity $0 = 0$ and a numerical contradiction such as $0 = 5$.
  – When both variables cancel during elimination or substitution, do not assume the system has no solution; check whether the remaining constant statement is true ($0 = 0$ in Example 1.3.3, indicating infinitely many solutions along coincident lines) or false ($0 = 5$ in Example 1.3.2, indicating no solutions between parallel lines).
- Compare both the slopes and the y-intercepts whenever two lines share the same slope.
  – Finding $m_1 = m_2$ only proves that the lines do not intersect at a single point; you must still compare b_1 and b_2 to determine whether the lines are strictly parallel ($b_1 \neq b_2$ in Example 1.3.2) or coincident ($b_1 = b_2$ in Example 1.3.3).
- Understand that a consistent-dependent system satisfies only the points on the shared line, not every point in $\mathbb{R}^2$.
  – Obtaining $0 = 0$ in Example 1.3.3 means the second equation adds no new restriction beyond the first, so every ordered pair (x, y) on the line $y = -\frac{1}{2}x + 3$ is a solution, whereas a point off the line such as $(0, 0)$ fails both equations ($2(0) + 4(0) \neq 12$).
- Divide every term on both sides by the coefficient of y when converting from standard form to slope-intercept form.
  – When isolating y in $6x - 2y = 3$ in Example 1.3.2, divide both $-6x$ and 3 by -2 to obtain $y = 3x - \frac{3}{2}$; forgetting to divide the constant term or dropping the negative sign corrupts the intercept comparison.