Vectors and Matrices

Unit 1 · Linear Systems and Matrix Algebra

Concept 1.3 · Interactive Reading Guide

Geometric Interpretation of Solutions

(Intersecting, Parallel, Coincident Lines)

In Concept 1.2, we solved two-variable linear systems using algebraic substitution and elimination. These notes connect those algebraic outcomes—a single coordinate pair, a false statement such as $0 = 5$, or an identity such as $0 = 0$—to the geometric behavior of lines in the Cartesian plane $\mathbb{R}^2$, classifying each system by its consistency and dependency. In Unit 2, we will transition from algebraic equations to matrix representations and systematic row reduction.

Learning Objective

Classify a system as consistent-independent, consistent-dependent, or inconsistent based on both algebraic outcomes (e.g., $0 = 0$ vs. $0 = 5$) and geometric line relationships.

Key Definitions

Consistent-Independent System.

Write the definition.

Inconsistent System.

Write the definition.

Consistent-Dependent System.

Write the definition.

1.3.1 Example: Intersecting Lines and Unique Solutions

> Determine the algebraic outcome and geometric line relationship for the following two-variable system in $\mathbb{R}^2$, and classify the system:

$$2x + y = 11$$
$$x - y = 1$$

Analyze the algebraic outcome and slope-intercept geometry.

Workspace: Analyze the algebraic outcome and slope-intercept geometry.

Classify the system.

Workspace: Classify the system.

1.3.2 Example: Parallel Lines and Algebraic Contradictions

> Analyze the following linear system both algebraically and geometrically, and classify the system:

$$3x - y = 4$$
$$6x - 2y = 3$$

Analyze the algebraic outcome and slope-intercept geometry.

Workspace: Analyze the algebraic outcome and slope-intercept geometry.

Classify the system.

Workspace: Classify the system.

1.3.3 Example: Coincident Lines and Redundant Constraints

Elena is calibrating two mixing recipes for a fruit base at a campus café. Let x represent the volume of concentrated juice (in fluid ounces) and let y represent the volume of water (in fluid ounces). The first recipe chart requires $2x+4y=12$, while a second production sheet requires $3x+6y=18$. Determine the algebraic and geometric relationship between these two constraints and classify the system:

$$2x+4y=12$$
$$3x+6y=18$$

Analyze the algebraic outcome and slope-intercept geometry.

Workspace: Analyze the algebraic outcome and slope-intercept geometry.

Classify the system.

Workspace: Classify the system.

1.3.4 The Geometric Classification Habit

When analyzing a mathematical system, translating between algebraic expressions and geometric graphs reveals why a system has one, none, or infinitely many solutions. In $\mathbb{R}^2$, each linear equation defines a straight line whose points (x, y) satisfy that equation, so the solution set of a two-variable system consists of every point that lies on all lines in the system simultaneously.

1. Reduce the system algebraically or rewrite each equation in slope-intercept form.
   - Use substitution or elimination to test whether the variables resolve to a specific pair (x, y) or vanish completely, and rewrite each equation as $y = mx + b$ (or $x = c$ for vertical lines) to identify the slope m and y-intercept b.
2. Interpret the resulting algebraic statement.
   - A specific coordinate pair (x, y) indicates a single shared solution; a false numerical statement such as $0 = 5$ indicates a contradiction with no solutions; a universally true identity such as $0 = 0$ indicates that the equations impose the exact same constraint.
3. Match the slope and intercept relationship to the line configuration.
   - Compare the slopes and intercepts: unequal slopes ($m_1 \neq m_2$) produce intersecting lines, equal slopes with distinct intercepts ($m_1 = m_2$ and $b_1 \neq b_2$) produce strictly parallel lines, and identical slopes with identical intercepts ($m_1 = m_2$ and $b_1 = b_2$) produce coincident lines.
4. State the complete two-part classification.
   - Pair the existence label (consistent or inconsistent) with the uniqueness label (independent or dependent) to classify the system as consistent-independent, inconsistent, or consistent-dependent.

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1.3.5 Geometric Classification Pitfalls

- Distinguish between the algebraic identity $0 = 0$ and a numerical contradiction such as $0 = 5$.
  - When both variables cancel during elimination or substitution, do not assume the system has no solution; check whether the remaining constant statement is true ($0 = 0$ in Example 1.3.3, indicating infinitely many solutions along coincident lines) or false ($0 = 5$ in Example 1.3.2, indicating no solutions between parallel lines).
- Compare both the slopes and the y-intercepts whenever two lines share the same slope.
  - Finding $m_1 = m_2$ only proves that the lines do not intersect at a single point; you must still compare b_1 and b_2 to determine whether the lines are strictly parallel ($b_1 \neq b_2$ in Example 1.3.2) or coincident ($b_1 = b_2$ in Example 1.3.3).
- Understand that a consistent-dependent system satisfies only the points on the shared line, not every point in $\mathbb{R}^2$.
  - Obtaining $0 = 0$ in Example 1.3.3 means the second equation adds no new restriction beyond the first, so every ordered pair (x, y) on the line $y = -\frac{1}{2}x + 3$ is a solution, whereas a point off the line such as $(0,0)$ fails both equations ($2(0) + 4(0) \neq 12$).
- Divide every term on both sides by the coefficient of y when converting from standard form to slope-intercept form.
  - When isolating y in $6x - 2y = 3$ in Example 1.3.2, divide both $-6x$ and 3 by -2 to obtain $y = 3x - \frac{3}{2}$; forgetting to divide the constant term or dropping the negative sign corrupts the intercept comparison.

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