Vectors and Matrices

Unit 1 · Linear Systems and Matrix Algebra

Topic 1 Introduction
Introduction to Linear Systems

Every day, engineers, operations planners, and scientists face decisions where multiple requirements must be satisfied at the same time. A manufacturing supervisor may need to schedule production runs so that several specialized departments are fully utilized without exceeding their weekly capacities, while a laboratory director may need to blend chemical stocks to hit exact concentration targets. Whenever several unknown quantities are tied together by multiple proportional relationships simultaneously, we cannot determine one unknown in isolation from the others. This topic introduces the foundational framework for translating those interconnected real-world requirements into precise mathematical relationships, finding the values that satisfy every condition at once, and interpreting why a set of requirements can yield a single exact answer, many possible answers, or no workable answer at all.

Topic 1 Overview

Across this topic, we examine simultaneous relationships from three complementary perspectives:

- **Concept 1.1 · Formulating Systems from Real-World Scenarios:** Translate verbal descriptions of resource limits, mixtures, and allocations into a structured collection of mathematical statements with clearly defined unknowns.
- **Concept 1.2 · Solving via Substitution & Recognizing Algebraic Limitations:** Determine exact values for two-unknown problems using direct hand-calculation techniques while observing why those manual approaches become unwieldy as problems grow larger.
- **Concept 1.3 · Geometric Interpretation of Solutions (Intersecting, Parallel, Coincident Lines):** Visualize two-unknown relationships in the plane to see how the physical arrangement of lines explains whether a problem has one solution, infinitely many solutions, or no solution.

Turning Everyday Constraints into Mathematical Structure (Concept 1.1)

Real-world planning problems rarely arrive ready for calculation; they begin as conversations, work orders, or operational reports filled with rates, unit requirements, and target totals. Before we can calculate a schedule or allocation, we need a reliable way to strip away the surrounding narrative and capture the exact quantitative rules governing the situation. If we leave our descriptions vague—such as confusing the number of batches produced with the number of hours each batch requires—any subsequent calculation will be meaningless.

In Concept 1.1, we focus strictly on this translation process. You will learn how to identify the specific unknown quantities a decision-maker needs to find, track how each unknown contributes to different totals or capacities, and align all of these conditions into a clean, consistent format. Establishing this disciplined structure at the outset ensures that no requirement is overlooked and prepares the problem for systematic analysis.

Finding Exact Values and Hitting a Computational Wall (Concept 1.2)

Once a scenario's requirements are clearly stated, the natural next step is to find the specific numbers that satisfy every condition at the same time. When only two unknown quantities are involved, familiar hand-algebra strategies allow us to express one unknown in terms of the other or combine conditions so that a single unknown can be pinned down first. For small problems, these direct algebraic techniques are intuitive and effective.

However, real industrial, scientific, and economic models rarely stop at two unknowns. In Concept 1.2, after mastering hand-solving strategies for two-unknown problems, you will see firsthand what happens when a third or fourth unknown is introduced alongside additional requirements. Because expressions shift shape at every step during manual back-and-forth replacements, keeping track of the arithmetic quickly becomes

error-prone and inefficient—motivating the need for the organized, grid-based methods we will develop in Topic 2.

Seeing Why Solutions Exist, Multiply, or Vanish (Concept 1.3)

Even in a simple two-unknown setting, not every set of requirements behaves nicely. Sometimes a manager's targets lock in a single, unmistakable production plan; other times, multiple different plans satisfy the exact same targets, or the requested targets turn out to be mutually contradictory and impossible to achieve together. When algebraic steps end in a surprising statement where all unknowns disappear, it helps to have a visual way to understand why.

Concept 1.3 connects the algebraic outcomes of a two-unknown problem to its visual geometry in the flat coordinate plane. Because each individual proportional requirement traces out a straight line of possibilities, a pair of requirements corresponds to two lines drawn on the same grid. Observing whether those two lines cross at a single point, overlap completely along the same path, or run side by side without ever meeting provides immediate visual insight into all three possible outcomes of a simultaneous problem.

Topic 1 Synthesis Problem: Precision Optics Production Scheduling

Keep the following scenario in mind as you work through Concepts 1.1, 1.2, and 1.3. We leave this problem unworked here so you can build each necessary skill across the topic before seeing its complete, step-by-step resolution in the **Topic 1 Conclusion**.

> Solstice Glassworks manufactures specialized optical components in weekly batches. The operations director is planning the weekly production schedule for two core product lines—Prism Lens batches and Optical Dome batches—across two bottleneck workshops: the Kiln-Annealing Studio and the Cold-Working Polishing Bay. To keep specialized equipment running at peak efficiency, the director wants the weekly production schedule to use the exact number of hours allocated to each workshop. The director evaluates four proposed scheduling scenarios for the upcoming month:
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> 1. **Schedule 1 (Baseline Two-Product Schedule):** Each batch of Prism Lenses requires 3 hours in the Kiln-Annealing Studio and 4 hours in the Cold-Working Polishing Bay. Each batch of Optical Domes requires 5 hours in the Kiln-Annealing Studio and 2 hours in the Cold-Working Polishing Bay. For this week, the director allocates exactly 43 hours in the Kiln-Annealing Studio and 34 hours in the Cold-Working Polishing Bay.
> 2. **Schedule 2 (Recalibrated Polishing Trial):** Kiln-Annealing requirements remain at 3 hours per Prism Lens batch and 5 hours per Optical Dome batch, with a weekly allocation of 45 hours. However, a new abrasive compound changes the Cold-Working Polishing Bay requirements to 6 hours per Prism Lens batch and 10 hours per Optical Dome batch, with a weekly allocation of 90 hours.
> 3. **Schedule 3 (Expanded Polishing Quota):** Using the same per-batch workshop times as Schedule 2 (3 hours and 5 hours in Kiln-Annealing; 6 hours and 10 hours in Cold-Working), the director keeps the Kiln-Annealing allocation at 45 hours but increases the Cold-Working Polishing Bay target to 100 hours.
> 4. **Schedule 4 (Three-Product Facility Expansion):** The studio introduces a third product line—Mirror Blank batches—and brings a third department online, the Ultrasonic Cleaning Tank. Under this expanded plan:
>   - In the Kiln-Annealing Studio, each Prism Lens batch requires 3 hours, each Optical Dome batch requires 5 hours, and each Mirror Blank batch requires 2 hours, with a total allocation of 37 hours.
>   - In the Cold-Working Polishing Bay, each Prism Lens batch requires 4 hours, each Optical Dome batch requires 2 hours, and each Mirror Blank batch requires 5 hours, with a total allocation of 47 hours.
>   - In the Ultrasonic Cleaning Tank, each Prism Lens batch requires 2 hours, each Optical Dome

batch requires 4 hours, and each Mirror Blank batch requires 3 hours, with a total allocation of 35 hours.

Questions to Consider

As the operations director reviews these four proposals to finalize the monthly production plan, consider the following operational questions:

(a) For each of the four schedules, what exact unknown batch counts is the director trying to determine, and how do the workshop hour requirements tie those batch counts to the weekly department allocations?

(b) Looking across Schedules 1, 2, and 3, which of these two-product schedules can actually be met, do any of them allow more than one valid combination of positive whole-number batches, and what specific batch counts work whenever a schedule is achievable?

(c) How does comparing the workshop trade-offs visually across Schedules 1, 2, and 3 explain why slight adjustments to polishing times or department targets completely change the director's scheduling options?

(d) In Schedule 4, what specific batch counts of Prism Lenses, Optical Domes, and Mirror Blanks fulfill all three department allocations, and why does working out this three-product schedule by hand become noticeably more tedious than analyzing the two-product schedules?